Chapter 4 – Part A
Motion in Two and Three Dimensions

4-1 Position and Displacement
4-2 Average Velocity and Instantaneous Velocity
4-3 Average Acceleration and Instantaneous Acceleration

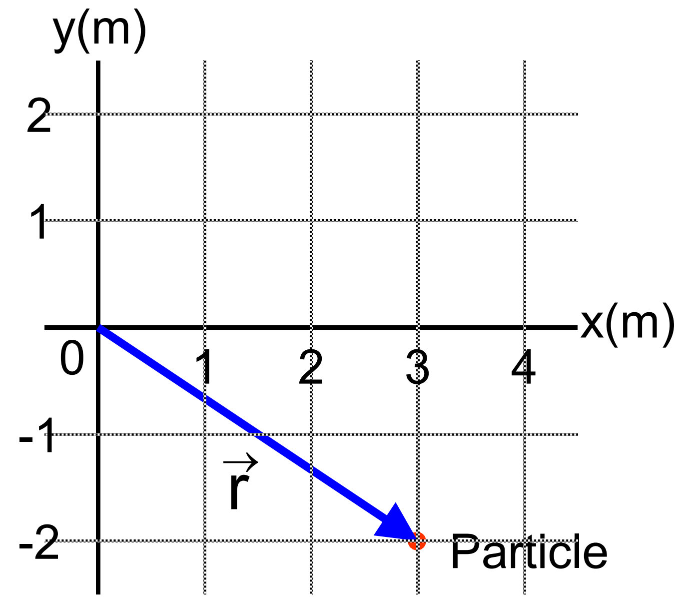
4-1 Position and Displacement
Position vector

The **position vector** $\vec{r}$ of a particle is a vector that extends from a reference point to the particle.

Usually the reference point is the origin of a coordinate system.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

x is the component of position vector $\vec{r}$ along the x axis



The particle has position vector

$$\vec{r} = (3.0 \text{ m})\hat{i} + (-2.0 \text{ m})\hat{j}$$

and rectangle coordinates

(3.0 m, -2.0 m, 0)

The particle has the rectangle coordinates (x, y, z)

4-1 Position and Displacement
Displacement

The **displacement** of a particle is the change in its position vector.

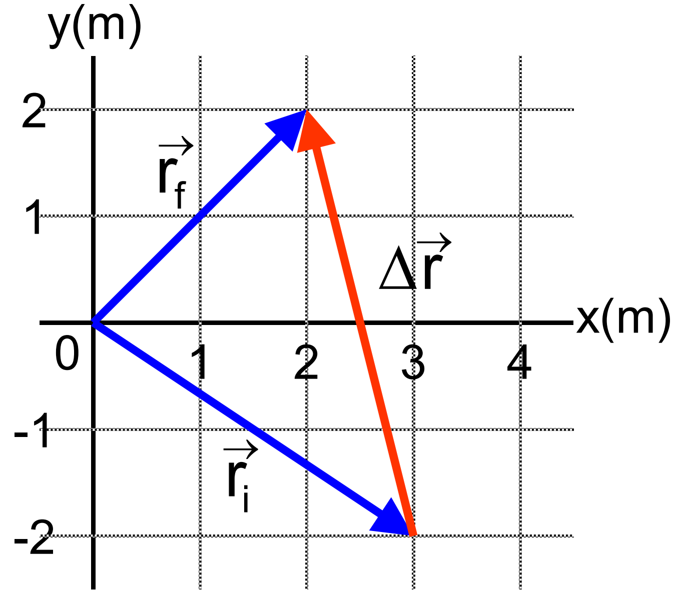
$$\text{Displacement} = \Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

$$\Delta\vec{r} = (x_f\hat{i} + y_f\hat{j} + z_f\hat{k}) - (x_i\hat{i} + y_i\hat{j} + z_i\hat{k})$$

$$\Delta\vec{r} = (x_f - x_i)\hat{i} + (y_f - y_i)\hat{j} + (z_f - z_i)\hat{k}$$

$$\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$$

Change in the x component of the position vector $\vec{r}$



$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

$$\vec{r}_f = \vec{r}_i + \Delta\vec{r}$$

$$\Delta\vec{r} = (2.0\text{ m} - 3.0\text{ m})\hat{i} + (2.0\text{ m} - (-2.0\text{ m}))\hat{j}$$

$$\Delta\vec{r} = (-1.0\text{ m})\hat{i} + (4.0\text{ m})\hat{j}$$

4-1 Position and Displacement
Example

The position vector for a particle is initially

$$\vec{r}_i = (-4.0\text{ m})\hat{i} - (1.0\text{ m})\hat{j} + (2.0\text{ m})\hat{k}$$

and then later is

$$\vec{r}_f = (3.0\text{ m})\hat{i} - (1.0\text{ m})\hat{j} + (3.0\text{ m})\hat{k}$$

What is the particle's displacement $\vec{r}$ from $\vec{r}_i$ to $\vec{r}_f$?

Solution

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

$$= [3.0 - (-4.0)]\hat{i} + [-1.0 - (-1.0)]\hat{j} + [3.0 - 2.0]\hat{k}$$

$$= (7.0\text{ m})\hat{i} + (1.0\text{ m})\hat{k}$$

4-1 Position and Displacement
Checkpoint

If a particle moves from xyz coordinates (-2 m, 4 m, -5 m) to coordinates (-4 m, -2 m, -5 m)

What is the displacement $\Delta\vec{r}$ in unit-vector notation?

Which coordinate planes is $\Delta\vec{r}$ parallel to?

Solution

$$\Delta\vec{r} = (-2.0\text{ m})\hat{i} + (-6.0\text{ m})\hat{j}$$

xy plane.

4-1 Position and Displacement

Example

A particle is moving in xy plane. The coordinates of its position are given by

$$x = -0.4\, t^2 + 2.5\, t - 1.0 \qquad \text{and} \qquad y = -0.3\, t^2 + 0.5\, t + 3.0,$$

where t in seconds and x and y in meters.

At t = 1 s, what is the particle's position vector in unit-vector notation and in magnitude-angle notation?

Solution

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

At t = 1 s,

$$x = -0.4\,(1)^2 + 2.5\,(1) - 1.0 = 1.1 \text{ m}$$

$$y = -0.3\,(1)^2 + 0.5\,(1) + 3.0 = 3.2 \text{ m}$$

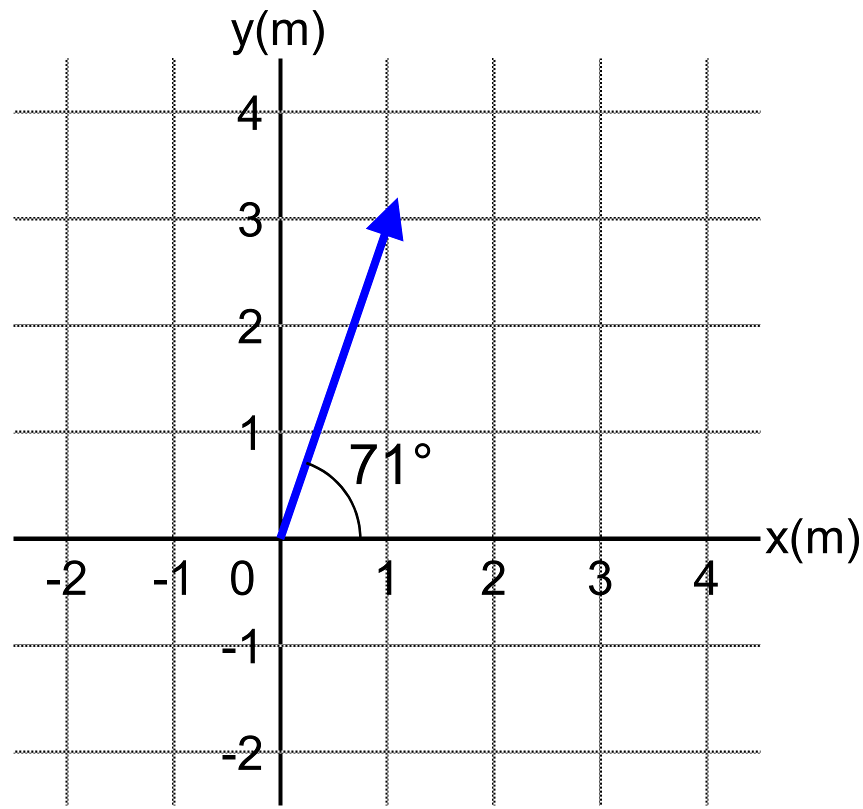
Unit-vector notation

$$\vec{r}(t) = (1.1 \text{ m})\hat{i} + (3.2 \text{ m})\hat{j}$$

Magnitude-angle notation

$$r = \sqrt{(1.1 \text{ m})^2 + (3.2 \text{ m})^2} = 3.4 \text{ m}$$

$$\theta = \tan^{-1} \frac{3.2 \text{ m}}{1.1 \text{ m}} = 71°$$



4-1 Position and Displacement

Example

A particle is moving in xy plane. The coordinates of its position are given by

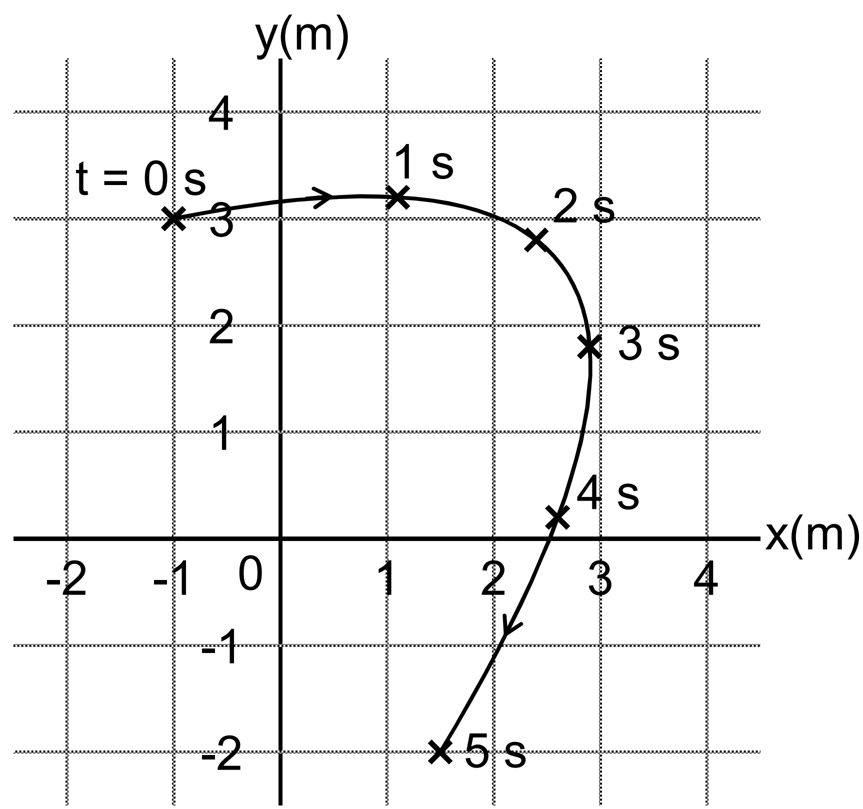
$$x = -0.4\,t^2 + 2.5\,t - 1.0 \qquad \text{and} \qquad y = -0.3\,t^2 + 0.5\,t + 3.0,$$

where t in seconds and x and y in meters.

Graph the particle's path for t = 0 to t = 5 s.

Solution

t(s)	x(m)	y(m)
0	-1.0	3.0
1	1.1	3.2
2	2.4	2.8
3	2.9	1.8
4	2.6	0.2
5	1.5	-2.0



4-2 Average Velocity and Instantaneous Velocity
Average velocity

The average velocity $\vec{v}_{avg}$ of a particle during a time interval is

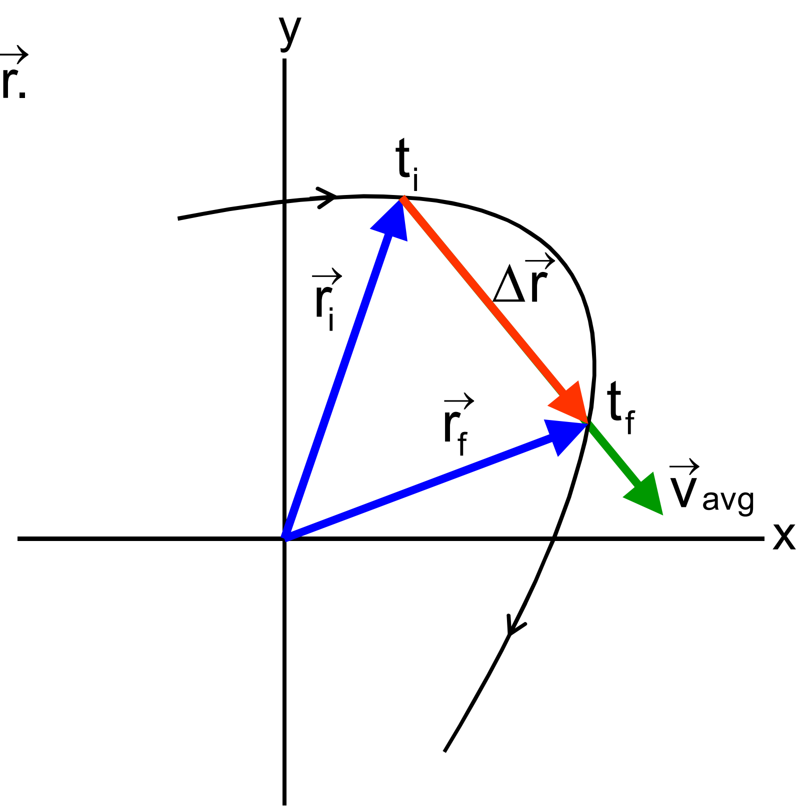
$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Time interval}}$$

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

$\vec{v}_{avg}$ has the same direction as that of $\Delta\vec{r}$.

$$\vec{v}_{avg} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t}$$

$$\vec{v}_{avg} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}$$

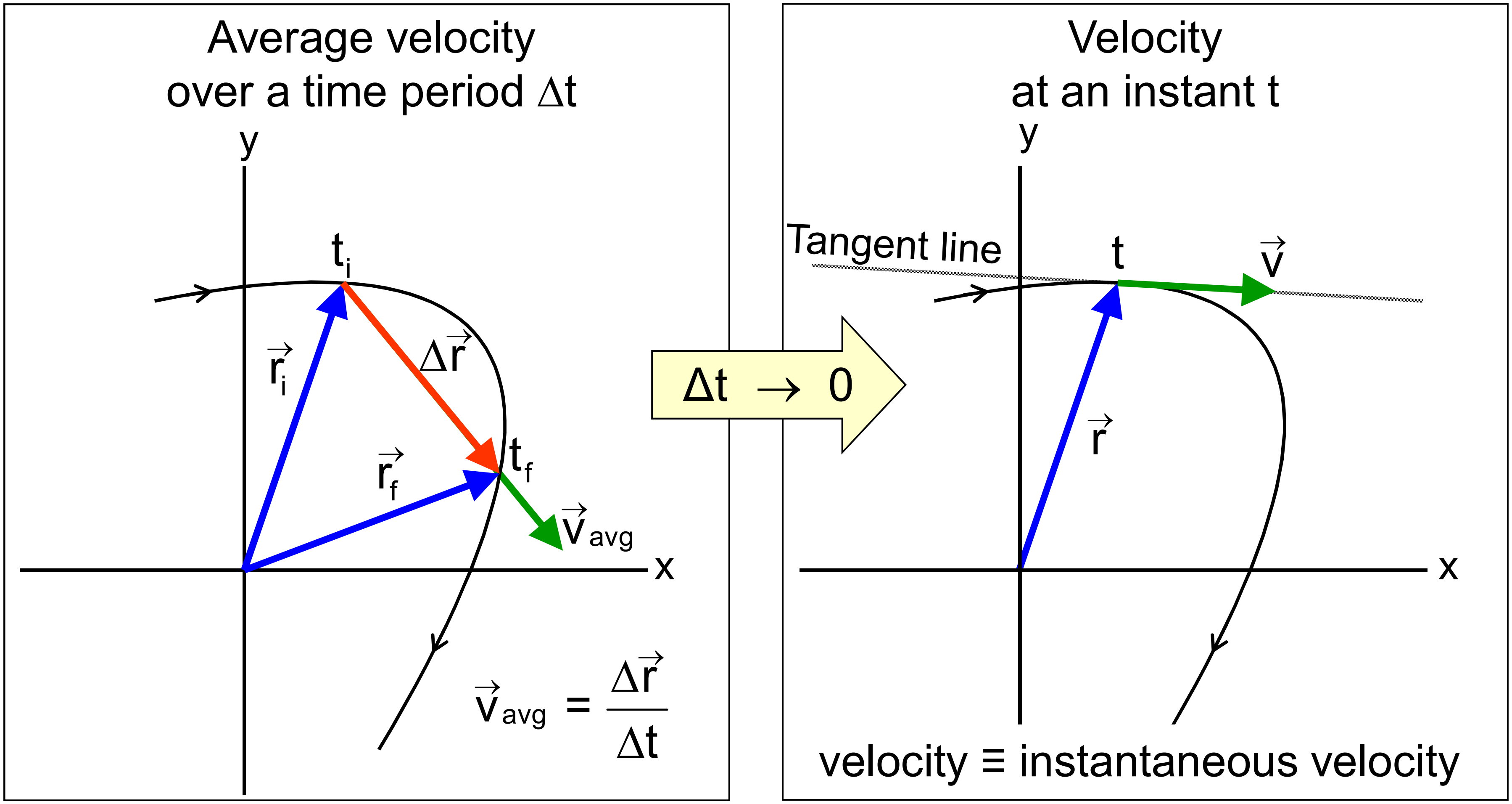


4-2 Average Velocity and Instantaneous Velocity

Instantaneous velocity

The velocity of a particle at some instant is

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$



4-2 Average Velocity and Instantaneous Velocity
Instantaneous velocity

The velocity of a particle at some instant is

$$\vec{v} = \frac{d\vec{r}}{dt}$$

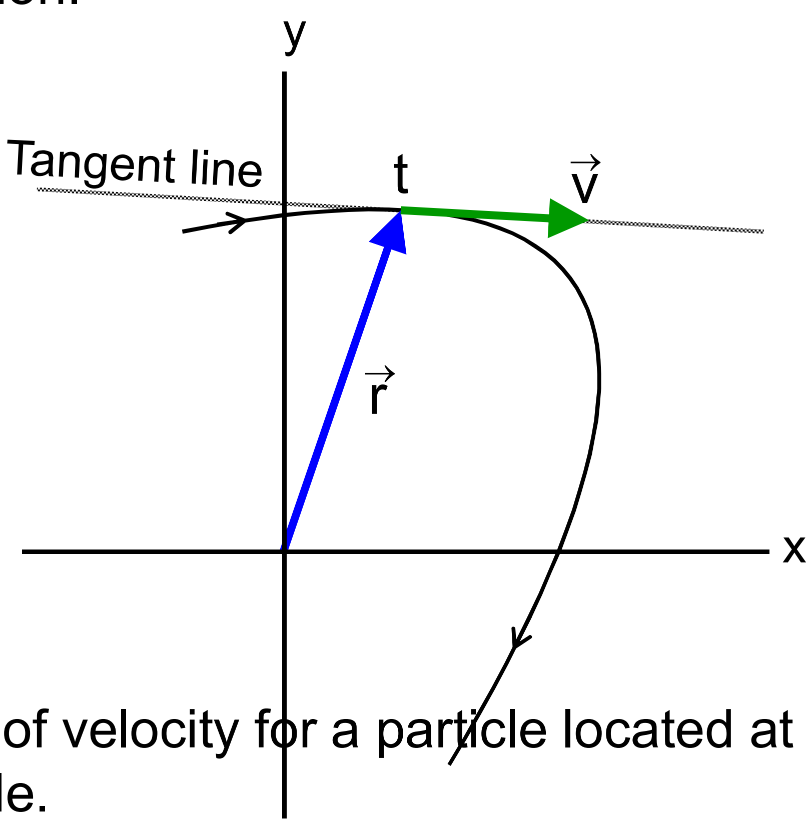
The direction of the instantaneous velocity of a particle is always tangent to the particle's path at the particle's position.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

the x component of velocity vector $\vec{v}$ along the x axis



The velocity vector shows the direction of velocity for a particle located at its tail. Its length can be drawn to any scale.

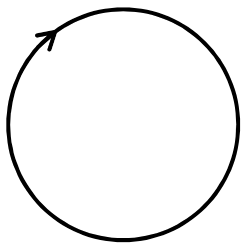
4-2 Average Velocity and Instantaneous Velocity

Checkpoint

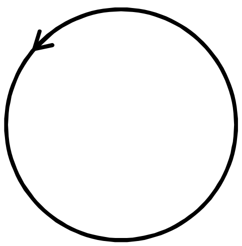
The figure shows a circular path taken by a particle. If

$$\vec{v} = (5.0\ \text{m/s})\hat{i} - (5.0\ \text{m/s})\hat{j}$$

Which quadrant is the particle moving at that instant if it is travelling

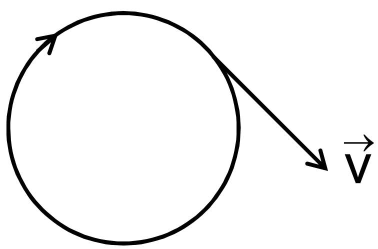


clockwise?

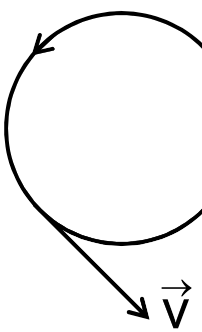


counterclockwise?

Solution



First quadrant



Third quadrant

4-2 Average Velocity and Instantaneous Velocity

Example

A particle is moving in xy plane. The coordinates of its position are given by

$$x = -0.4\,t^2 + 2.5\,t - 1.0 \qquad \text{and} \qquad y = -0.3\,t^2 + 0.5\,t + 3.0,$$

where t in seconds and x and y in meters.

What is the particle's velocity at t = 2 s?

Solution

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(-0.4\,t^2 + 2.5\,t - 1.0)$$

$$= -0.8\,t + 2.5 = -0.8\,(2) + 2.5 = 0.9 \text{ m/s}$$

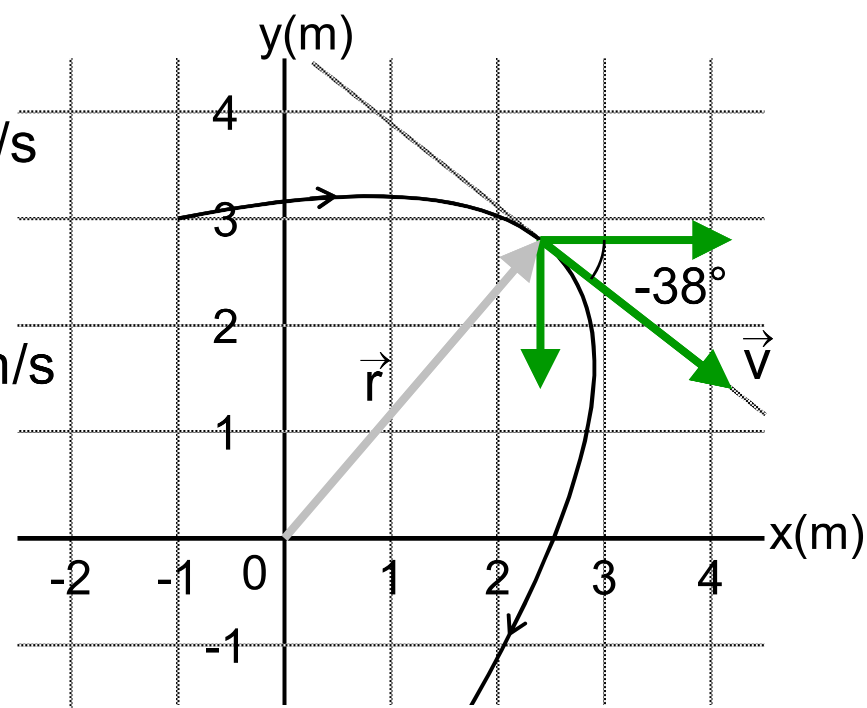
$$v_y = \frac{dy}{dt} = \frac{d}{dt}(-0.3\,t^2 + 0.5\,t - 3.0)$$

$$= -0.6\,t + 0.5 = -0.6\,(2) + 0.5 = -0.7 \text{ m/s}$$

$$\vec{v} = (0.9 \text{ m/s})\hat{i} + (-0.7 \text{ m/s})\hat{j}$$

$$v = \sqrt{(0.9 \text{ m/s})^2 + (-0.7 \text{ m/s})^2} = 1.1 \text{ m/s}$$

$$\theta = \tan^{-1}\frac{-0.7 \text{ m/s}}{0.9 \text{ m/s}} = -38^\circ$$



At t = 2s, $\vec{r} = (2.4 \text{ m})\hat{i} + (2.8 \text{ m})\hat{j}$

4-3 Average Acceleration and Instantaneous Acceleration

Average acceleration

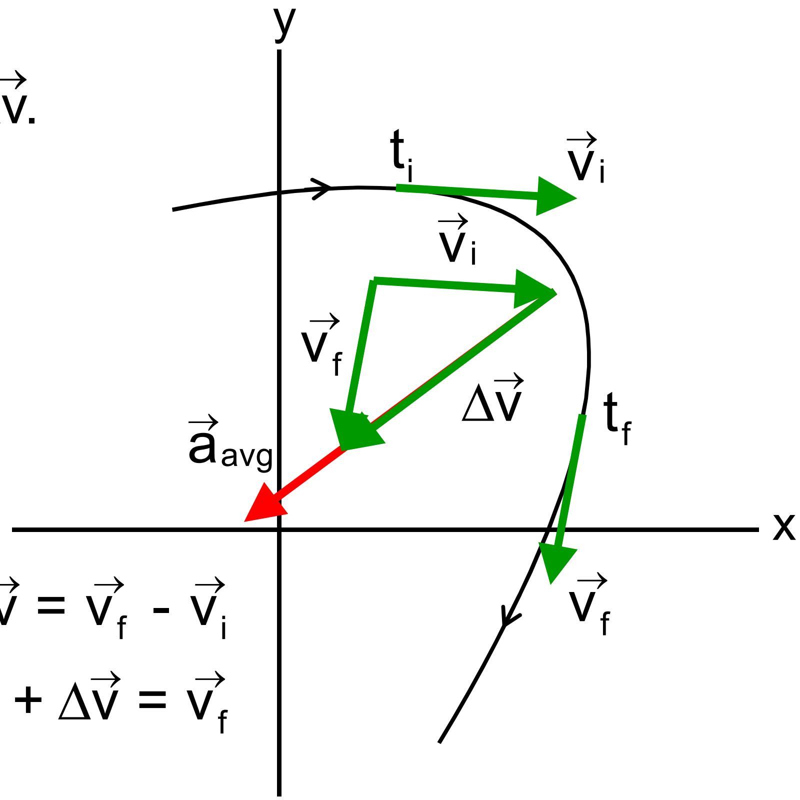
The average acceleration $\vec{a}_{avg}$ of a particle during a time interval is

$$\text{Average acceleration} = \frac{\text{Change in velocity}}{\text{Time interval}}$$

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

$\vec{a}_{avg}$ has the same direction as that of $\Delta\vec{v}$.

Since multiplying a vector quantity ($\Delta\vec{v}$) by a positive scalar quantity ($1/\Delta t$) changes only the magnitude of the vector, not its direction.



$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$

$\vec{v}_i + \Delta\vec{v} = \vec{v}_f$

4-3 Average Acceleration and Instantaneous Acceleration

Instantaneous acceleration

The acceleration $\vec{a}$ of a particle at some instant is

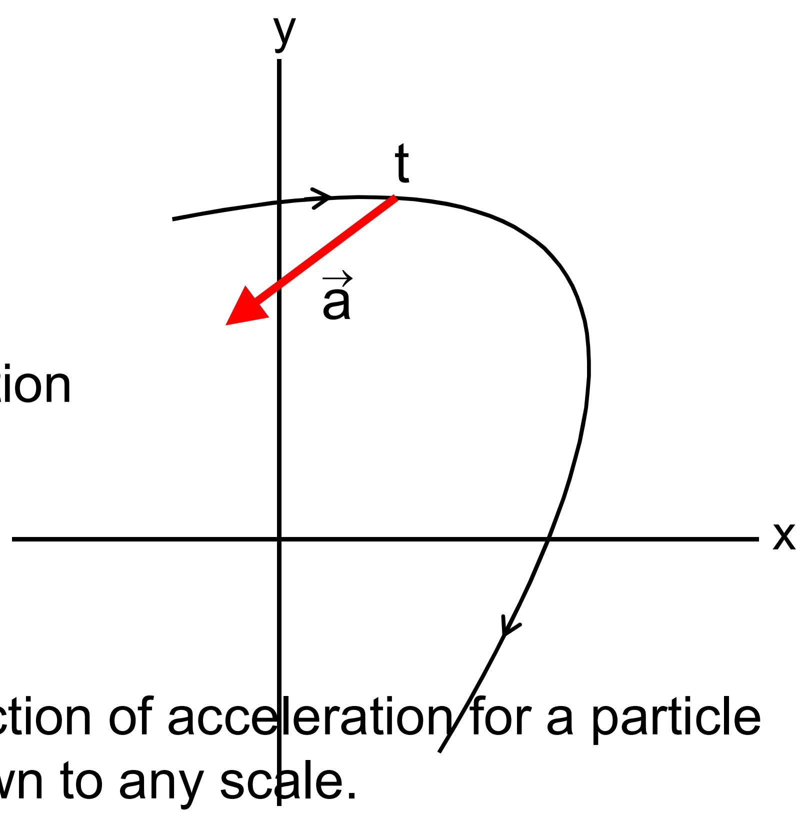
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$

$$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

the x component of acceleration vector $\vec{a}$ along the x axis



The acceleration vector shows the direction of acceleration for a particle located at its tail. Its length can be drawn to any scale.

4-3 Average Acceleration and Instantaneous Acceleration

Example

A particle is moving in xy plane. The coordinates of its position are given by

$x = -0.4\,t^2 + 2.5\,t - 1.0$ and $y = -0.3\,t^2 + 0.5\,t + 3.0$,

where t in seconds and x and y in meters.

What is the particle's acceleration at t = 2 s?

Solution

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{d}{dt}x\right) = \frac{d^2}{dt^2}x = -0.8\ \text{m/s}^2$$

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}\left(\frac{d}{dt}y\right) = \frac{d^2}{dt^2}y = -0.6\ \text{m/s}^2$$

$$\vec{a} = (-0.8\ \text{m/s}^2)\hat{i} + (-0.6\ \text{m/s}^2)\hat{j}$$

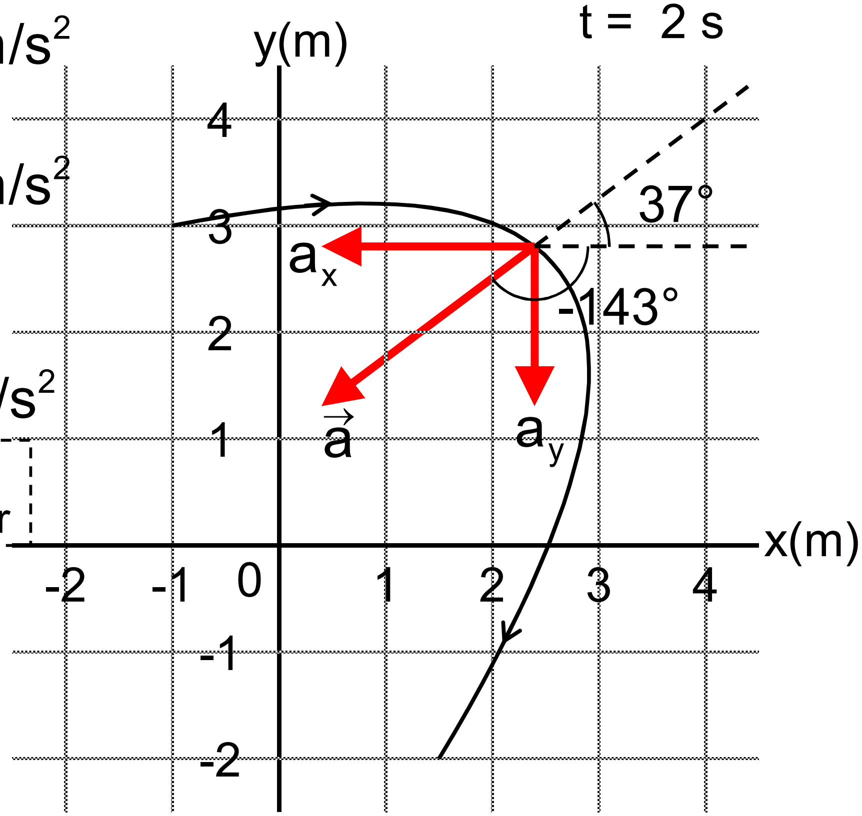
$$a = \sqrt{(-0.8\ \text{m/s}^2)^2 + (-0.6\ \text{m/s}^2)^2} = 1.\ \text{m/s}^2$$

$$\theta = \tan^{-1}\frac{-0.6\ \text{m/s}}{-0.8\ \text{m/s}} \boxed{\cancel{= 37^\circ}}$$

Using a calculator

This angle is not consistent with the directions of the acceleration.
The correct answer is

$$\theta = 37^\circ - 180^\circ = -143^\circ$$



4-3 Average Acceleration and Instantaneous Acceleration
Motion in two dimensions

Motion in two dimensions is equivalent to
two independent one-dimensional motions:
one in the x direction and one in the y direction.

For constant acceleration,

motion in two dimensions

$$\vec{v} = \vec{v}_0 + \vec{a}\, t$$

$$\vec{r} - \vec{r}_0 = \vec{v}_0\, t + \frac{1}{2}\vec{a}\, t^2$$

is equivalent to two independent
one-dimensional motions.

motion parallel the x direction

$$v_x = v_{0x} + a_x\, t$$

$$x - x_0 = v_{0x}\, t + \frac{1}{2} a_x\, t^2$$

motion parallel the y direction

$$v_y = v_{0y} + a_y\, t$$

$$y - y_0 = v_{0y}\, t + \frac{1}{2} a_y\, t^2$$

4-3 Average Acceleration and Instantaneous Acceleration

Checkpoint

The following are equations describing the position of a particle moving in xy plane. Determine whether the acceleration is constant.

$x = -6\,t^2 + 5\,t - 2$ and $y = -3\,t^2 - 2\,t$

$x = -6\,t^3 + 5\,t$ and $y = -2\,t^2 + 4$

$\vec{r} = 2\,t^2\hat{i} - (4\,t + 3)\hat{j}$

$\vec{r} = (3\,t^3 - 2\,t)\hat{i} - 3\,\hat{j}$

Solution

$$a_x = \frac{d^2}{dt^2}x$$

$$a_y = \frac{d^2}{dt^2}y$$

Constant acceleration

Not constant acceleration

Constant acceleration

Not constant acceleration

4-3 Average Acceleration and Instantaneous Acceleration

Example

A particle has a constant acceleration,

$a = 4.0\ m/s^2$ at 120^0 from the + x axis.

At t = 0, the particle's velocity is $\vec{v}_0 = (-2.0\ m/s)\hat{i} + (3.0\ m/s)\hat{j}$

What is the particle's velocity at t = 6.0 s?

Solution

Motion in two dimensions is equivalent to two independent one-dimensional motions: one in the x direction and one in the y direction.

$$a_x = a\cos\theta = (4.0\ m/s^2)\cos 120° = -2.0\ m/s^2$$
$$a_y = a\sin\theta = (4.0\ m/s^2)\sin 120° = 3.5\ m/s^2$$

<u>Motion in the x direction</u>

$$v_x = v_{0x} + a_x t$$
$$= (-2.0\ m/s) + (-2.0\ m/s^2)(6.0\ s) = -14\ m/s$$

<u>Motion in the y direction</u>

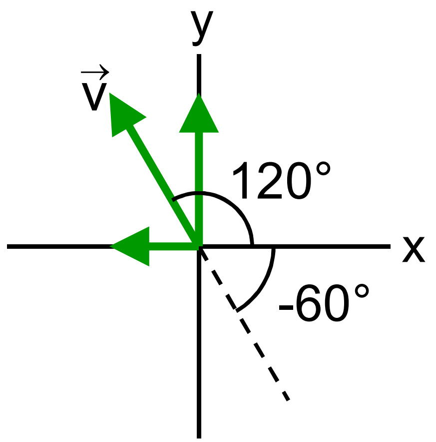
$$v_y = v_{0y} + a_y t$$
$$= (3.0\ m/s) + (3.5\ m/s^2)(6.0\ s) = 24\ m/s$$

$$\vec{v} = (-14\ m/s)\hat{i} + (24\ m/s)\hat{j}$$

$$v = \sqrt{(-14\ m/s)^2 + (24\ m/s)^2} = 28\ m/s$$

$$\theta = \tan^{-1}\frac{24\ m/s}{-14\ m/s} \boxed{= \cancel{-60°}}$$

Using a calculator



$$\theta = -60° + 180° = 120°$$

4-3 Average Acceleration and Instantaneous Acceleration
Checkpoint

If the position of a particle is given by

$$\vec{r} = (3t^3 - 2t)\hat{i} - 6\hat{j},$$

with r in meters and t in seconds. What must be the units of the coefficients 3, -2, and -6?

Solution

$$[\vec{r}] = [3t^3\hat{i}] = [-2t\hat{i}] = [-6\hat{j}]$$

Since $[\hat{i}] = [\hat{j}] = 1$,

$$[\vec{r}] = [3t^3] = [-2t] = [-6]$$

The unit of the coefficient 3 is m/s^3
The unit of the coefficient -2 is m/s
The unit of the coefficient -6 is m

4-3 Average Acceleration and Instantaneous Acceleration
Checkpoint

1. A particle is moving in xy plane. The coordinates of its position are given by $x = -0.4\ t^2 + 2.5\ t - 1$ and $y = -0.3\ t^2 + 0.5\ t + 3$, where t in seconds and x and y in meters. What is the particle's velocity at t = 2 s?

2. A particle is moving in xy plane. The coordinates of its position are given by $x = 4\ t^2 + 5\ t - 2$ and $y = 4\ t^2 + 3\ t + 4$, where t in seconds and x and y in meters. What is the particle's velocity at t = 4 s?

3. A particle has a constant acceleration, $a = 4.0\ m/s^2$ at 120^0 from the + x axis.
At t = 0, the particle's velocity is $\vec{v}_0 = (3.0\ m/s)\hat{i} + (-2.0\ m/s)\hat{j}$

What is the particle's velocity at t = 6.0 s?

4-3 Average Acceleration and Instantaneous Acceleration

Checkpoint

1. A particle is moving in xy plane. The coordinates of its position are given by $x = -0.4\,t^2 + 2.5\,t - 1$ and $y = -0.3\,t^2 + 0.5\,t + 3$, where t in seconds and x and y in meters. What is the particle's velocity at t = 2 s?

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What is the particle's velocity at t = 6.0 s?

4-3 Average Acceleration and Instantaneous Acceleration
Checkpoint

1. A particle is moving in xy plane. The coordinates of its position are given by $x = -0.4\,t^2 + 2.5\,t - 1$ and $y = -0.3\,t^2 + 0.5\,t + 3$, where t in seconds and x and y in meters. What is the particle's velocity at t = 2 s?

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