Chapter 2 – Part B
Vectors

2- 6 Multiplying Vectors – Scalar Product
2- 7 Multiplying Vectors – Vector Product

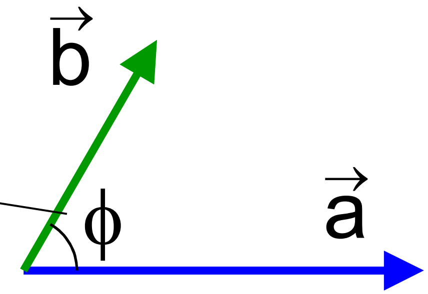
2-6 Multiplying Vectors – Scalar Product

Scalar Product

$$\vec{a}\bullet\vec{b} = a\,b\cos\phi$$

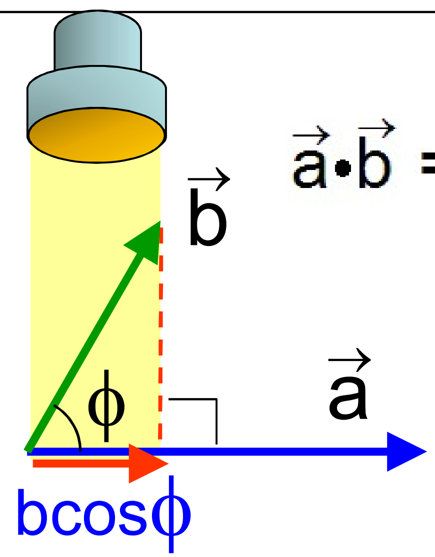
The result is a Scalar quantity.

Angle between the two vectors

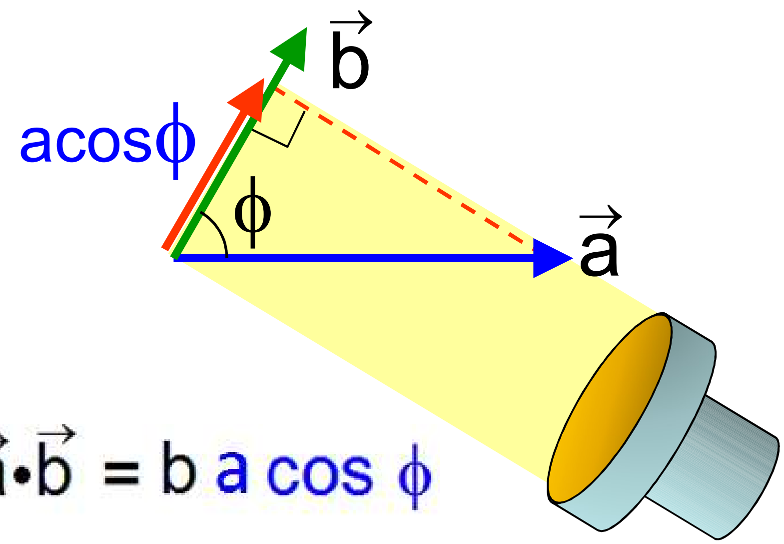


Scalar product is also called dot product and read as a dot b.

For dot product use cosine. Dot →CO



$$\vec{a}\bullet\vec{b} = a\,b\cos\phi$$



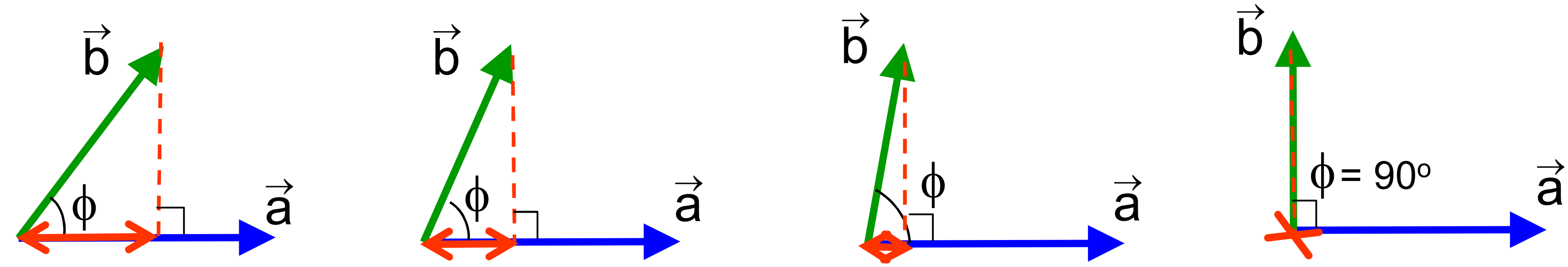
$$\vec{a}\bullet\vec{b} = b\,a\cos\phi$$

The component of $\vec{b}$ along the direction of $\vec{a}$ is $b\cos\phi$

The component of $\vec{a}$ along the direction of $\vec{b}$ is $a\cos\phi$

2-6 Multiplying Vectors – Scalar Product

Scalar Product



As the lines become perpendicular the projection gets shorter and becomes zero when the angle is 90^o.

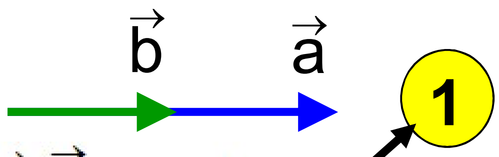
- The dot product of two vectors is a scalar.
- It is largest if the two vectors are parallel, and zero if the two vectors are perpendicular.

- One of the common applications of the **scalar(dot) product** is to find the angle between two vectors.

2-6 Multiplying Vectors – Scalar Product

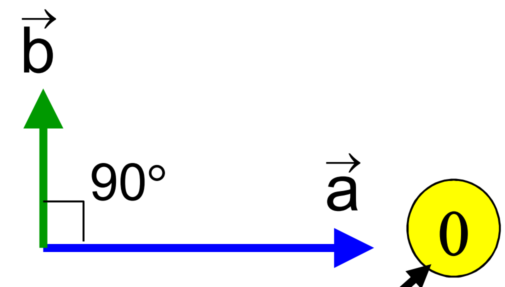
Scalar Product

$$\vec{a}\cdot\vec{b} = a\,b\cos\phi$$



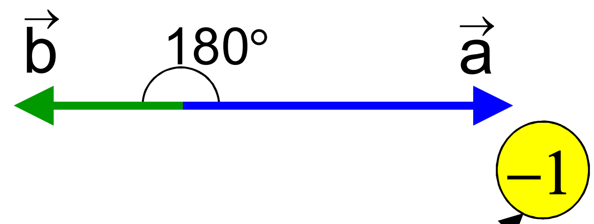
$\vec{a}\cdot\vec{b} = a\,b\cos 0$

$\vec{a}\cdot\vec{b} = a\,b$



$\vec{a}\cdot\vec{b} = a\,b\cos 90$

$\vec{a}\cdot\vec{b} = 0$



$\vec{a}\cdot\vec{b} = a\,b\cos 180$

$\vec{a}\cdot\vec{b} = -\,a\,b$

$\hat{i}\cdot\hat{i} = 1$

$\hat{j}\cdot\hat{j} = 1$

$\hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = 0$

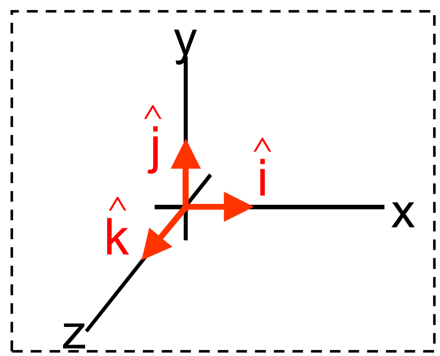
$\hat{i}\cdot\hat{k} = 0$

$\hat{k}\cdot\hat{j} = 0$

$$\vec{a}\cdot\vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k})\cdot(b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Magnitude of a vector: $a=\sqrt{a^2} = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$



2-6 Multiplying Vectors – Scalar Product
Example 1

Vector a has magnitude 4, vector b has magnitude 6 and the angle between a and b is 90°. Calculate $\vec{a}\cdot\vec{b}$.

Solution

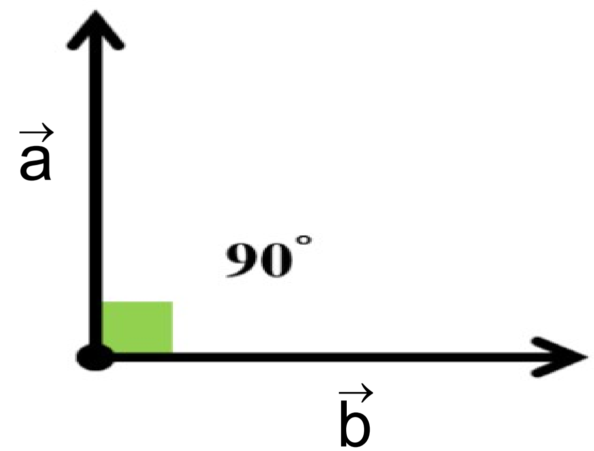
$$\vec{a}\cdot\vec{b} = a\ b\cos\phi$$

$$|\vec{a}| = 4$$

$$|\vec{b}| = 6$$

$$\vec{a}\cdot\vec{b} = a.b\cos 90 \quad (\cos 90 \to 0)$$

$$\vec{a}\cdot\vec{b} = 4.6.0 = 0$$

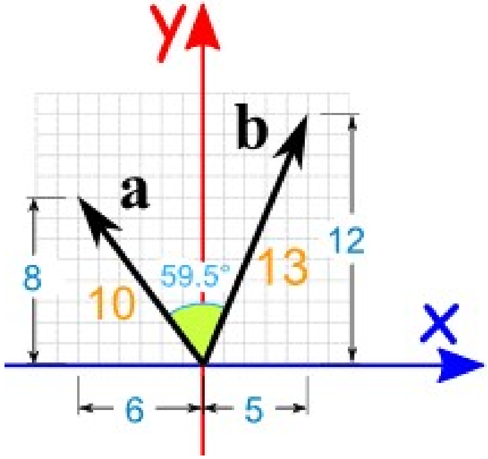


2-6 Multiplying Vectors – Scalar Product

Example 2

Look at the graph and calculate the value of $\vec{a} \bullet \vec{b}$.

The values of a = 10, and b = 13.



Solution I: $\vec{a} \bullet \vec{b} = a\ b \cos \phi$

$\vec{a} \bullet \vec{b} = 10 \times 13 \times \cos(59.5) = 130 \times (0.505)$

$\vec{a} \bullet \vec{b} = 66$

Solution II: $a_x = -6 \quad b_x = 5 \quad a_y = 8 \quad b_y = 12$

$\vec{a} \bullet \vec{b} = a_x \, . \, b_x + a_y \, . \, b_y = -6.5 + 8.12$

$\vec{a} \bullet \vec{b} = 66$

2-6 Multiplying Vectors – Scalar Product

Example 3

What is the angle between $\vec{a} = 3.0\hat{i} - 4.0\hat{j}$ and $\vec{b} = -2.0\hat{i} + 3.0\hat{k}$?

Solution

$$\vec{a} \bullet \vec{b} = a\, b \cos \phi$$

$$\cos \phi = \frac{\vec{a} \bullet \vec{b}}{a\, b}$$

$$\vec{a} \bullet \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{a} \bullet \vec{b} = (3.0)(-2.0) + (-4.0)(0) + (0)(3.0) = -6.0$$

$$a = \sqrt{3.0^2 + (-4.0)^2} = 5.0$$

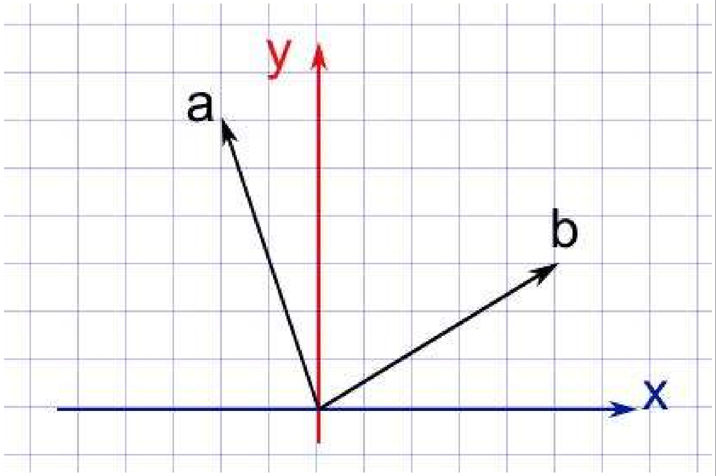
$$b = \sqrt{(-2.0)^2 + (3.0)^2} = 3.6$$

$$\phi = \cos^{-1} \frac{-6.0}{(5.0)(3.6)} = 110^\circ$$

2-6 Multiplying Vectors – Scalar Product

Example 4

Look at the graph and calculate the value of $\vec{a} \bullet \vec{b}$.



Solution

$$\vec{a} = -2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$$

$$\vec{b} = 5\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}$$

$$\vec{a} \bullet \vec{b} = a_x \,.\, b_x + a_y \,.\, b_y$$

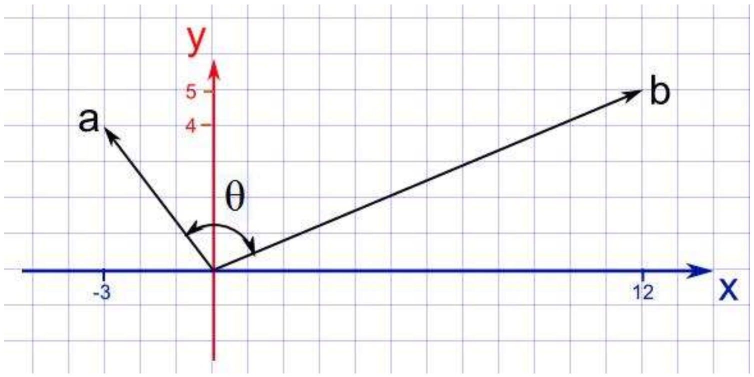
$$\vec{a} \bullet \vec{b} = a_x \,.\, b_x + a_y \,.\, b_y = -2.5 + 6.3$$

$$\vec{a} \bullet \vec{b} = 8$$

2-6 Multiplying Vectors – Scalar Product

Example 5

Use the dot product and calculate the angle θ.



Solution

$\vec{a} = -3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$ $\quad |a| = \sqrt{(3)^2 + (4)^2} = 5$

$\vec{b} = 12\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}$ $\quad |b| = \sqrt{(12)^2 + (5)^2} = 13$

$|a|.|b| = 5.13 = 65$

$\vec{a} \bullet \vec{b} = a_x . b_x + a_y . b_y \qquad a_x = -3 ;\ b_x = 12;\ a_y = 4;\ b_y = 5$

$\vec{a} \bullet \vec{b} = a_x . b_x + a_y . b_y = -3.12 + 4.5 = -16$

$$\cos\theta = \frac{\vec{a} \bullet \vec{b}}{a\ b} = \frac{-16}{65} = -0.18$$

$\theta = 104.3^{o}$

2-6 Multiplying Vectors – Scalar Product

Example 6

Two vectors are given as: **a** = -3.0 i + 5.0 j + 4.0 k and **b** = 4.0 i + 5.0 j + 3.0 k, where i,j and k are the unit vectors in the positive x, y and z directions. Find the angle between the vectors A and B.

Solution

a = -3.0 i + 5.0 j + 4.0 k

b = 4.0 i + 5.0 j + 3.0 k

$$|a| = \sqrt{(-3)^2 + (5)^2 + (4)^2} = 7.1$$

$$|\mathbf{b}| = \sqrt{(4)^2 + (5)^2 + (3)^2} = 7.1$$

$$|a| \,.\, |b| = 7.1x7.1 = 50.4$$

$a_x = -3$; $b_x = 5$; $a_y = 5$; $b_y = 5$; $a_z = 4$; $b_z = 3$

$$\vec{a}\bullet\vec{b} = a_x \,.\, b_x + a_y \,.\, b_y + a_z \,.\, b_z$$

$$\vec{a}\bullet\vec{b} = -3.5 + 5.5 + 4.3 = 22$$

$$\cos\theta = \frac{\vec{a}\bullet\vec{b}}{a\ b} = \frac{22}{50.4} = 0.44$$

$$\theta = 64^o$$

2-6 Multiplying Vectors – Scalar Product

Example 7

Two vectors **a** and **b** have magnitudes of 10 m and 15 m respectively. The angle between them is 65°. Find the component (projection) of **b** along **a**.

Solution

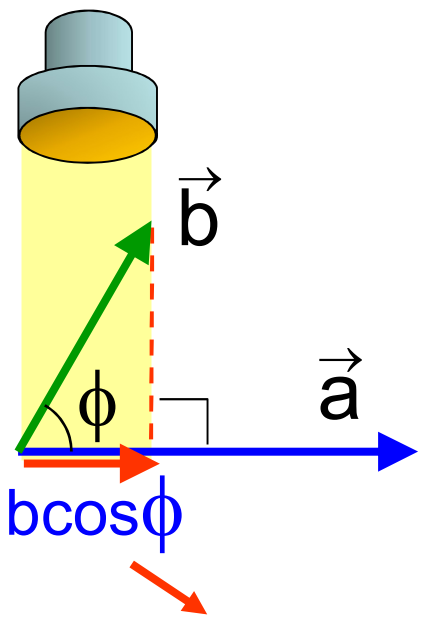
a = 10 m and b = 15

$\phi = 65^{o}$

The projection of b along is bcos ϕ.

Find bcos ϕ.

b.cos ϕ = 15.cos 65 = 6.3 m



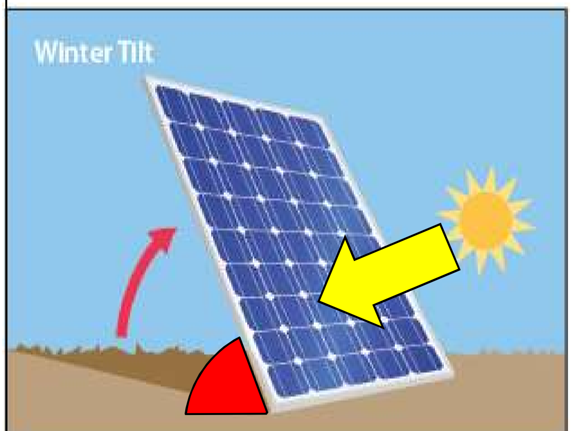
The projection of b along is bcos ϕ.

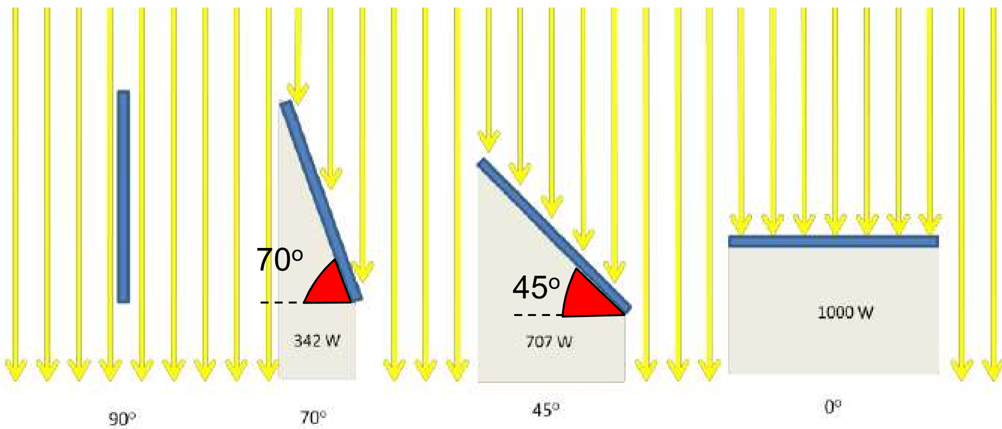
2-6 Multiplying Vectors – Scalar Product

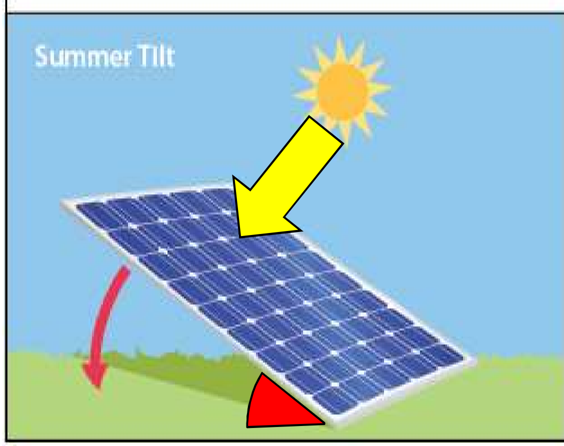
Why are solar panels usually mounted on an angle?

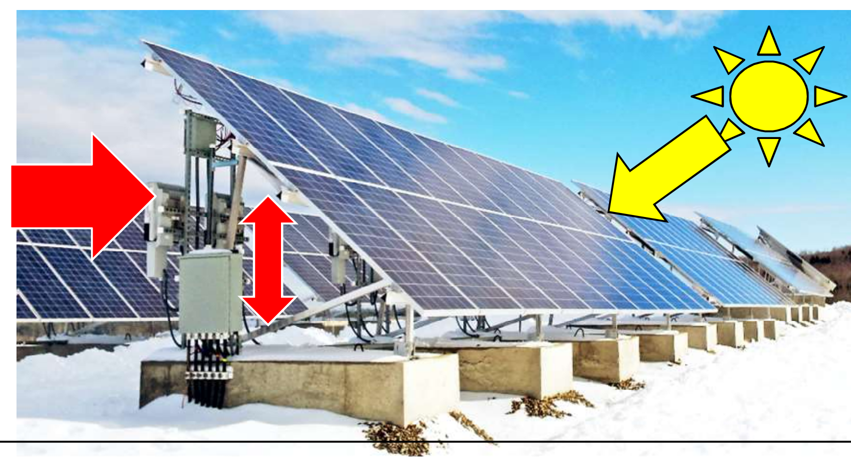
Solar Panel alignment and Scalar multiplication

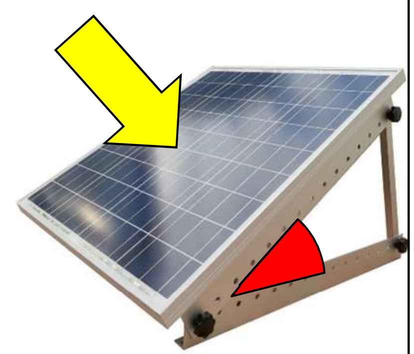
To get optimum benefit from sunlight, the solar panel should face the sun properly such that the maximum amount of sunlight will hit the panel surface. Alignment of the panel effects the amount of sun light it receives and the electricity it produces. Depending on the position of the sun, tilt angle of the panel should be changed to increase the efficiency.







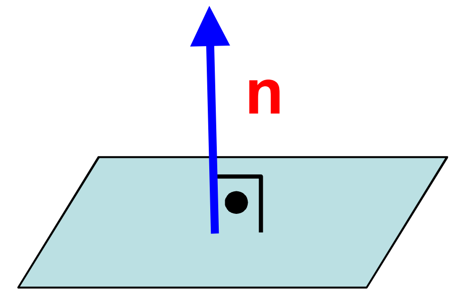


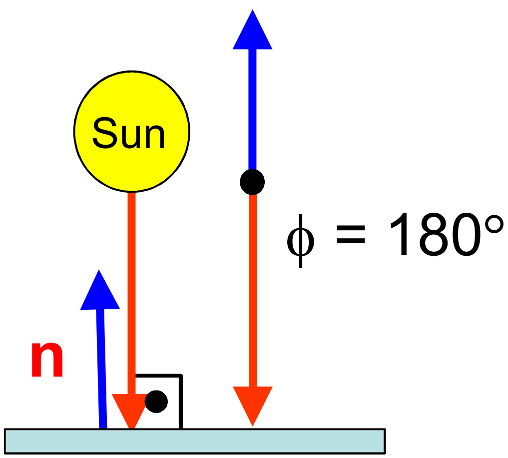


2-6 Multiplying Vectors – Scalar Product

Why are solar panels usually mounted on an angle?

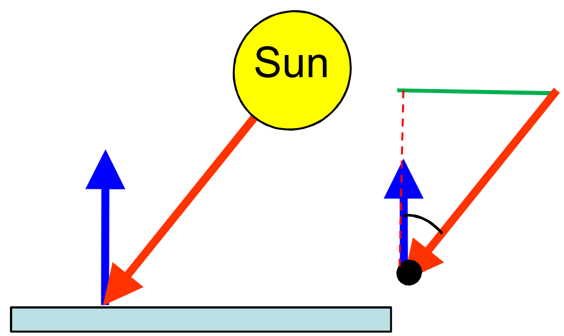
The solar panel has a surface, the tilt angle of the panel changes the normal vector of the panel surface. The angle between the sunlight and the normal vector of the surface effects the amount of solar energy captured. The normal vector, often simply called the "normal," to a surface is a vector which is perpendicular to the surface at a given point.





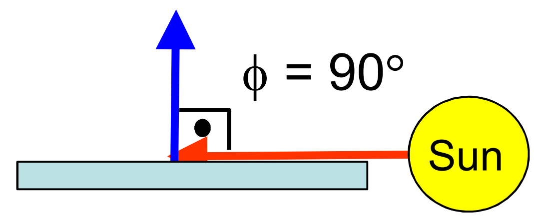
Solar panel is more efficient when properly facing the sun such that the sun light hits on the panel surface vertically. The effect of the sun light is calculated by cos180º(=-1).

$$\vec{a}\cdot\vec{b} = -\,a\,b$$



Solar panel is less efficient when it is not properly facing the sun such that the sun light hits on the panel with an angle. The effect of the sun light is calculated by cosϕ.

$$\vec{a}\cdot\vec{b} = a\,b\cos\phi$$



If the sun light is parallel to the solar panel, there will be no efficiency of the panel. The effect of the sun light is calculated by cos90º(=0)

$$\vec{a}\cdot\vec{b} = 0$$

2-6 Multiplying Vectors – Scalar Product
Example 8

Two vectors have magnitudes of 5 m and 3 m respectively. What is the angle between them if their dot product is

(a) zero

(b) 15 m^2

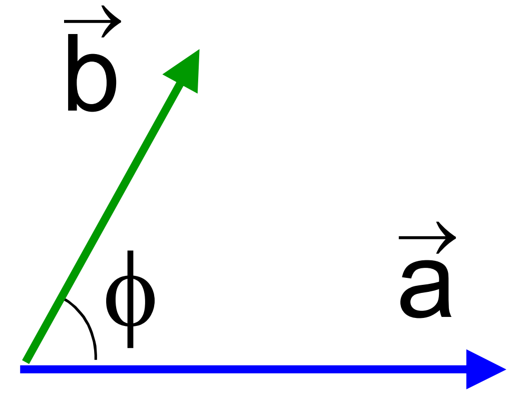
(c) -15 m^2

Solution

$\phi = 90°$

$\phi = 0°$

$\phi = 180°$



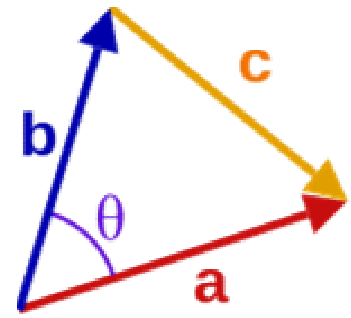
$$\vec{a} \bullet \vec{b} = a\, b \cos \phi$$

2-6 Multiplying Vectors – Scalar Product

Example 9

Using dot product equation prove the law of cosine using the vectors for the triangle beside.

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



$\mathbf{c} = \mathbf{a} - \mathbf{b}$

Solution

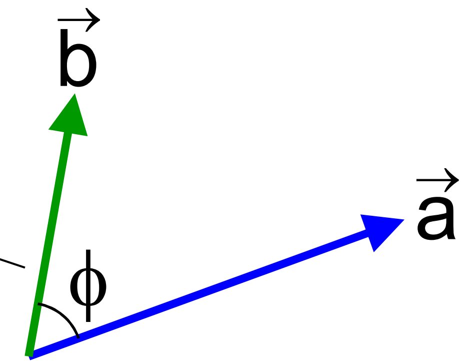
$$\begin{aligned}
\mathbf{c}\cdot\mathbf{c} &= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\
&= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \\
&= a^2 - \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} + b^2 \\
&= a^2 - 2\mathbf{a}\cdot\mathbf{b} + b^2 \\
c^2 &= a^2 + b^2 - 2ab\cos\theta
\end{aligned}$$

2-7 Multiplying Vectors - Vector Product

Vector Product or Cross Product

$$\vec{a} \times \vec{b} = a\,b\,\sin\phi$$

ϕ is the small angle between $\vec{a}$ and $\vec{b}$.



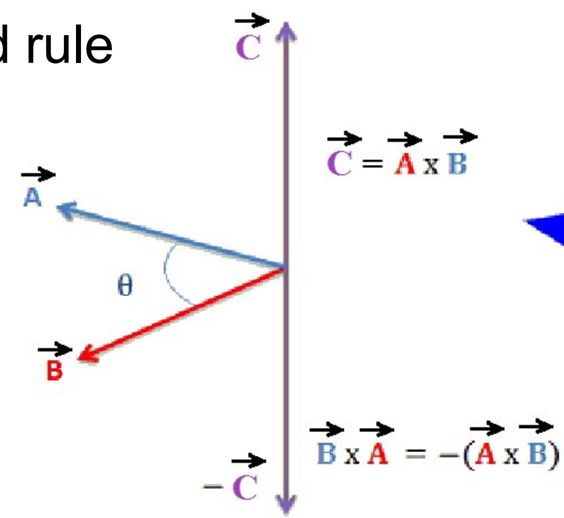
Vector product is also called cross product and read as a cross b.

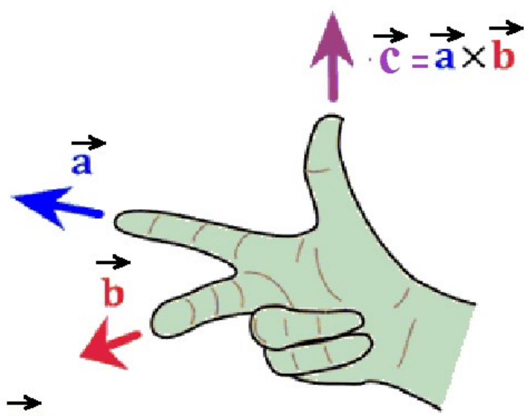
For cross product use sine. CrosS →Sine

The direction of $\vec{c}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$

If $\vec{a}$ and $\vec{b}$ are in the plane of the paper, $\vec{c}$ will be perpendicular to the paper .

Right-hand rule





right hand

Your thumb of your right hand points along the direction of the cross product c if your index finger points along the direction of the first vector a and your middle finger points along the second vector b.

2-7 Multiplying Vectors - Vector Product

Example 10

Let the magnitude of vector **a** is 2 and **b** is 5, and the angle between them is 30^o**.** Find **a** x **b.**

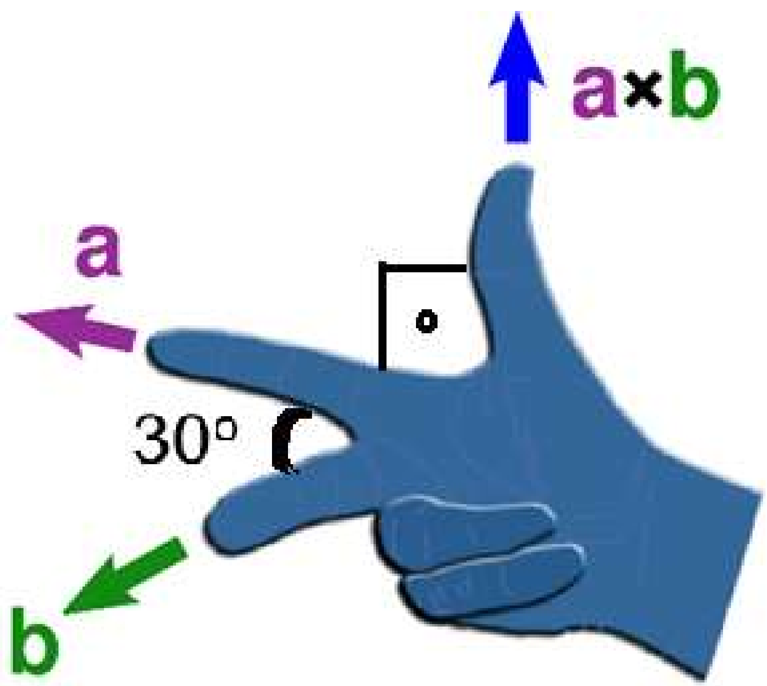
Solution

a = 2 and b = 5.

$$\vec{a} \times \vec{b} = a\ b\ \sin\phi$$

$$\vec{a} \times \vec{b} = 2 \cdot 5\ \sin 30^o$$

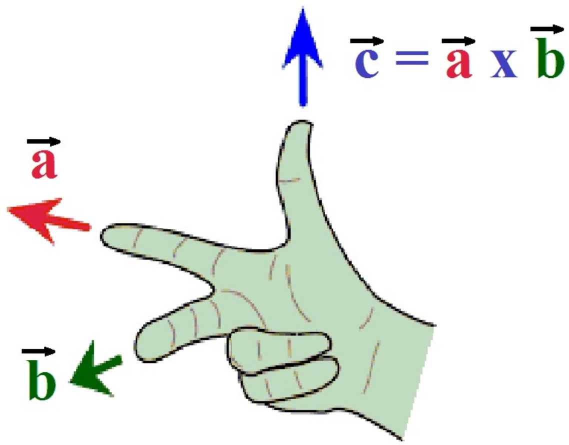
$$\vec{a} \times \vec{b} = 10\ .\ (0.5) = 5\,\hat{k}$$



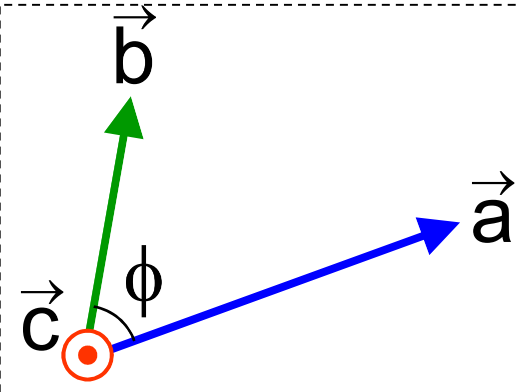
2-7 Multiplying Vectors - Vector Product

Vector Product

Right-hand rule

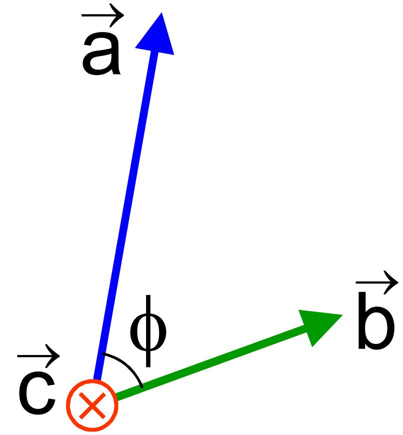


$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$



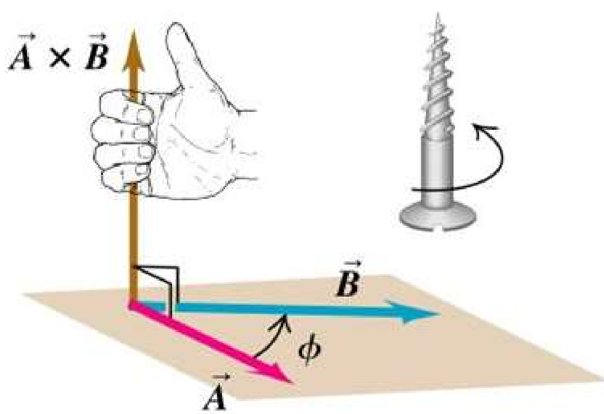
Out of the page

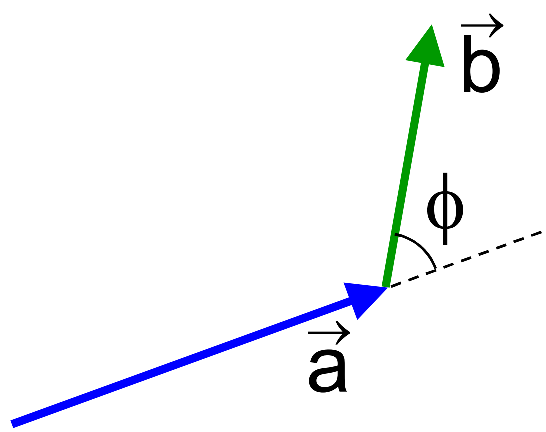
an arrow comes out of the page

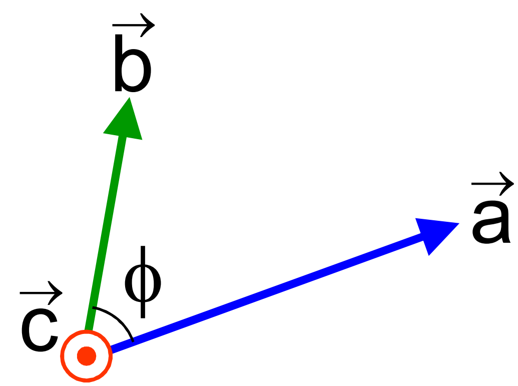


Into of the page

an arrow goes into the page





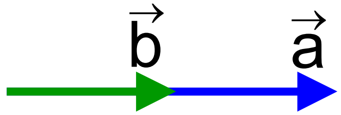


2-7 Multiplying Vectors - Vector Product

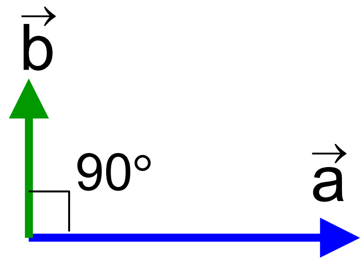
Vector Product

$$\vec{a} \times \vec{b} = \vec{c}$$

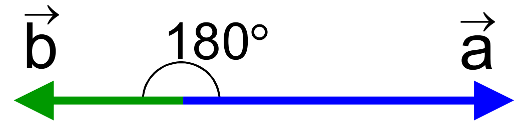
$$c = a\,b \sin\phi$$



$c = 0$



$c = a\,b$

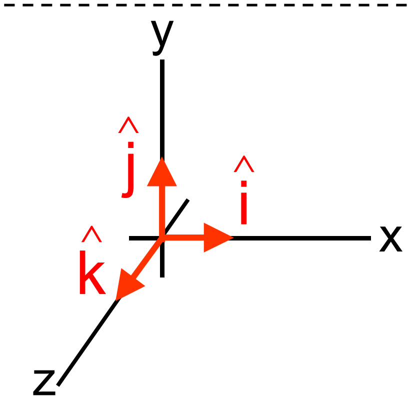


$c = 0$

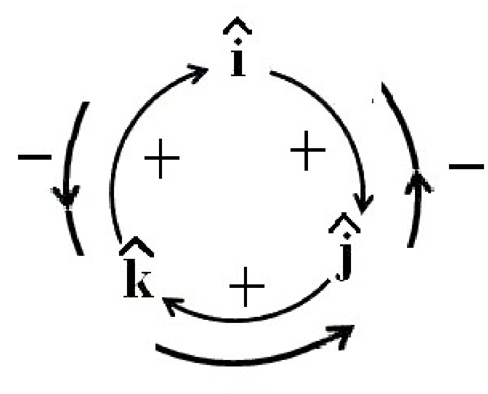
$$\hat{i} \times \hat{i} = 0$$
$$\hat{j} \times \hat{j} = 0$$
$$\hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{k} \times \hat{j} = -\hat{i}$$
$$\hat{i} \times \hat{k} = -\hat{j}$$



Note the order.
i always comes before j.
j always comes before k.
k always comes before i.
If they are not in this order,
then the answer is negative.



2-7 Multiplying Vectors - Vector Product

Vector Product in unit-vector notation

$$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

$\hat{i} \times \hat{i} = 0$	$\hat{i} \times \hat{j} = \hat{k}$
$\hat{j} \times \hat{j} = 0$	$\hat{j} \times \hat{k} = \hat{i}$
$\hat{k} \times \hat{k} = 0$	$\hat{k} \times \hat{i} = \hat{j}$

Dot product: your result is a number.

For dot product use cosine. Dot →CO

Cross product: your result is a vector.

For cross product use sine. CrosS →Sine

2-7 Multiplying Vectors - Vector Product
Example 11

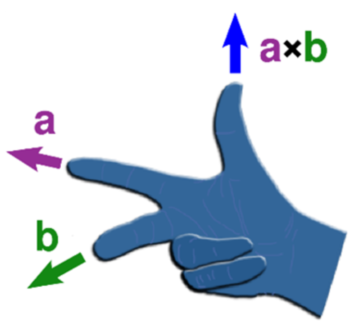
What is $\vec{c} = \vec{a} \times \vec{b}$ if $\vec{a} = 3\hat{i} - 4\hat{j}$ and $\vec{b} = -2\hat{i} + 3\hat{k}$?

Solution

$$\vec{c} = \vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

$$\vec{c} = ((-4)(3)-(0)(0))\hat{i} + ((0)(-2)-(3)(3))\hat{j} + ((3)(0)-(-4)(-2))\hat{k}$$

$$\vec{c} = -12\hat{i} - 9\hat{j} - 8\hat{k}$$



Note that $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

We check that by showing $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$.

$$\vec{c} \cdot \vec{a} = c_x a_x + c_y a_y + c_z a_z$$

$$\vec{c} \cdot \vec{a} = (-12)(3) + (-9)(-4) + (-8)(0) = 0$$

$$\vec{c} \cdot \vec{b} = c_x b_x + c_y b_y + c_z b_z$$

$$\vec{c} \cdot \vec{b} = (-12)(-2) + (0)(-4) + (-8)(3) = 0$$

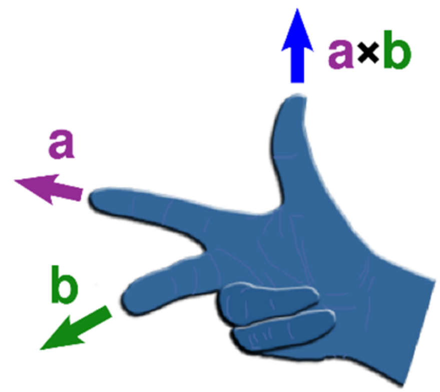
2-7 Multiplying Vectors - Vector Product

Example 12

Let **a = -2i + 3k** and **b = 2j + k.** Find **a x b.**

Solution

$a_x = 2; a_y = 0; a_z = 3; b_x = 0; b_y = 2; b_z = 1$



$$\vec{c} = \vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

$$\mathbf{c} = (0.1 - 3.2)\,\mathbf{i} + (3.0 - 2.1)\,\mathbf{j} + (2.2 - 0.0)\,\mathbf{k}$$

$$\mathbf{c} = -6\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k}$$